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Continuous parametric families of stationary and translating periodic point vortex configurations

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The number of periodic arrangements of point vortices – point vortex streets – in two-dimensional fluid flow that are stationary is known to be finite for a generic choice of vortex circulations. When all circulations are the same in absolute value, however, stationary vortex street configurations have been associated with the zeros of certain trigonometric polynomials containing free complex parameters. The presence of these parameters may prove useful in constructing point vortex models of shear layers and wakes. In this paper it is shown that such a continuum of stationary configurations exists in a much wider class of point vortex street systems. The circulations may take on many values, not just two, providing increased flexibility in the modelling context. A simple method for computing these configurations is derived. The effects of symmetries on the solution polynomials are described, and illustrated with examples. In addition, novel translating vortex street configurations are found having arbitrary translation velocity and containing free parameters for vortex circulations ± 1 and also for vortex circulations $+1$, -2.

1. Introduction

Stationary configurations of point vortices have been a subject of investigation for over a century (Aref *et al.* 2003). Besides their intrinsic interest, they figure in the dynamics of point vortices as critical points of the vortex interaction energy. Periodic configurations of point vortices, or point vortex streets, form simple finite-dimensional models of shear layers, jets and wakes in a two-dimensional ideal fluid. In this context, the stationary periodic configurations represent steady or quasi-steady fluid motions. Two classical examples are a single row of identical point vortices modelling a shear layer, and the Kármán vortex street model of the wake behind a bluff body. Both configurations are unstable, but can persist for surprisingly long times before succumbing to unstable modes. More generally, vortex streets containing point vortices of varying strengths can model shear layers with some initial internal structure, or the more 'exotic' wakes found behind oscillating bluff bodies (Aref, Stremler & Ponta 2006) or the multiple oscillating airfoils encountered in insect flight (Wang 2005). The shear layer models use stationary periodic configurations, while the wake models involve configurations in which all vortices move uniformly in the periodic direction.

All stationary and translating configurations containing two or three vortices per period have been described in detail (Stremler 2003), but very few such configurations are known when the number n of vortices per period is greater than 3. It is known that there are $(n-1)!$ stationary configurations for generic choice of vortex circulations (O'Neil 1987, 2006).

Non-periodic systems of point vortices in which all circulations are the same in absolute value have a characteristic that differs markedly from the generic case. Namely, stationary configurations are formed by the zeros of polynomials that have free complex parameters, so that the set of configurations is a continuum rather than finite. These polynomials satisfy certain differential equations. The first to write down a differential equation in the context of vortices was Tkachenko (1964), although this equation had been analysed much earlier (Burchnall & Chaundy 1930). Further work can be found in Bartman (1983), Kadtke & Campbell (1987), Loutsenko (2003, 2004), O'Neil (2006). Recently these solutions were extended to a parametrized family of trigonometric polynomials, creating stationary periodic point vortex configurations (Loutsenko 2003). In the context of flow models, the presence of free parameters gives greatly increased flexibility in matching the model system to the non-singular fluid flow of interest.

In this paper it is shown that these versatile continua of stationary configurations exist in a much wider class of point vortex street systems. Some are quite simple, e.g. the configuration with vortices of circulation -1 at locations n and $n + 1/2$ in the complex plane, and of circulation 2 at $n + z$ and $n - z$ for every integer n; this configuration is stationary for all non-zero values of the complex parameter z. When the imaginary part of z is large, this gives the appearance of three distinct shear layers, while small values cause the layers to merge in various ways. Other configurations exist with arbitrarily large numbers of vortices in each period. The vortex circulations may take on many different values, the chief restrictions being that all must be of the same sign except one, and that they must be half-integral multiples of the exceptional circulation. Thus a wide variety of new periodic point vortex configurations, all with a free parameter, are available to create heterogeneous (rather than homogeneous, identical-vortex) models of shear layers.

The next two sections of this paper introduce the necessary background and notation, as well as the main tool used in this work: a differential equation for polynomials representing vortex street configurations, equation (3.1) below (O'Neil 2006). Use of this differential equation in § 4 allows known closed-form expressions for *stationary* configurations of vortex streets of circulation ± 1 to be modified, creating new families of *translating* configurations with arbitrary common velocity. Section 5 establishes sufficient conditions for using one solution to (3.1) to create an independent second solution which can be combined with the first to yield a solution with a free complex parameter. An effective method for computing these solutions is then derived, and examples presented. The analysis shows that each family of stationary configurations transforms to a configuration that translates uniformly, i.e. all vortices have the same velocity, for a limiting value of the complex parameter. Thus translating configurations, such as those described in § 4, form a convenient starting point for the computation. The effect of symmetries on the solution polynomials is described in § 6. Finally, a novel set of translating periodic configurations with circulations 1 and -2 and arbitrary translation velocity are described in § 7. These configurations also have a free complex parameter and form a continuum, so that the corresponding stationary configurations have two free complex parameters. The last section summarizes the results obtained and suggests a few directions for further work.

2. An algebraic system of equations for periodic point vortex equilibria

Given a complex number z and a (complex) period L, the simplest periodic configuration of point vortices consists of a point vortex of (real) circulation Γ at location

$z+mL$ in the complex plane for every integer $m \in \boldsymbol{Z}$. The position z is only meaningful modulo $L\boldsymbol{Z}$ so we may use the 'complex cylinder coordinate' $u = \exp(2\pi i z/L)$ to describe this configuration without ambiguity. Note that addition of a constant to all vortex positions corresponds to multiplication of u by a non-zero constant, and $z \to i\infty$ corresponds to $u \to 0$.

More generally, a periodic arrangement of point vortices is described by vortex circulations Γ_j, vortex positions $z_j + L\boldsymbol{Z}$ and cylinder coordinates $u_j = \exp(2\pi i z_j/L)$, $1 \leqslant j \leqslant n$. The velocity v_j of the vortex of circulation Γ_j at position $z_j + mL$ is independent of m, so that the dynamics preserves the periodicity of the vortex array:

$$2\pi i\, \bar{v}_j = \sum_{k:k\neq j} \frac{\Gamma_k}{L}\ \pi \cot \pi L^{-1}(z_j - z_k), \tag{2.1}$$

where the bar represents complex conjugation; see e.g. (Aref *et al.* 2003). Because the vortices at positions $z_j + L\boldsymbol{Z}$ move in unison, they will be considered to be a single entity termed a *point vortex street*; thus equations (2.1) describe the dynamics of a system of n point vortex streets. (This non-standard terminology greatly simplifies much of the subsequent discussion; alternatively one may describe a street as a single point vortex on the cylinder $\boldsymbol{C}/L\boldsymbol{Z}$.) By a simple change of scale, we set $L = 1$ and supress it from subsequent formulae. In terms of the coordinates u_j equation (2.1) takes the form

$$2\bar{v}_j = \sum_{k:k\neq j} \Gamma_k \frac{u_j + u_k}{u_j - u_k}. \tag{2.2}$$

The right-hand side of (2.2) is homogeneous in the u_j, corresponding to the invariance of (2.1) under translation $z_j \mapsto z_j + c$. Note too that equations (2.1) and (2.2) are invariant under the two involutions $z_j \mapsto -\bar{z}_j$, $u_j \mapsto \bar{u}_j$, $v_j \mapsto \bar{v}_j$ and $z_j \mapsto \bar{z}_j$, $u_j \mapsto 1/\bar{u}_j$, $v_j \mapsto -\bar{v}_j$, as well as their composition $z_j \mapsto -z_j$, $u_j \mapsto 1/u_j$, $v_j \mapsto -v_j$. These involutions correspond to reflections in the y-axis, the x-axis and the origin respectively.

A *translating* point vortex street configuration has $v_j = v \neq 0$ in (2.2) for all j. Since multiplying (2.1) or (2.2) by Γ_j and summing shows $\sum_j \Gamma_j v_j = 0$, it follows that a translating configuration must be *neutral*, i.e. the total circulation $S := \sum_j \Gamma_j$ is zero. A *stationary* configuration has all $v_j = 0$ and may be non-neutral (have non-zero S.)

Assume that $u_1, \ldots, u_n$ are distinct complex numbers and u is a complex variable. Using the identity

$$\frac{u_j + u_k}{(u - u_j)(u - u_k)} = \frac{u_j + u_k}{u_j - u_k}\left(\frac{1}{u - u_j} - \frac{1}{u - u_k}\right)$$

and equation (2.2), it is easy to verify the relation

$$\frac{1}{2}\sum_{j\neq k} \Gamma_j \Gamma_k \frac{u_j + u_k}{(u - u_j)(u - u_k)} = \sum_j \frac{\Gamma_j}{u - u_j}(2\bar{v}_j). \tag{2.3}$$

Consequently a translating or stationary configuration will satisfy:

$$\frac{1}{2}\sum_{j\neq k} \Gamma_j \Gamma_k \frac{u_j + u_k}{(u - u_j)(u - u_k)} - 2\bar{v}\sum_j \frac{\Gamma_j}{u - u_j} = 0. \tag{2.4}$$

This relation can hold even when the u_j are not all distinct, and (2.2) is singular. Multiplication of (2.4) by $(u - u_1)\cdots(u - u_n)$ gives a polynomial in u on the left-hand side, the coefficients of which are homogeneous polynomials in the u_j. For example

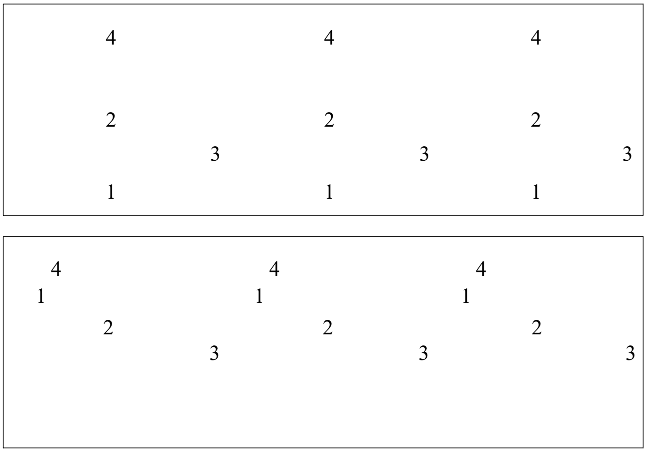

FIGURE 1. Stationary vortex street configurations for circulations $\Gamma_1 = -2/3$, $\Gamma_2 = -2$, $\Gamma_3 = 7/2$, $\Gamma_4 = 1$. The configurations have period 1, and a portion of the plane containing three periods is depicted. Vortices with circulation Γ_1 are denoted by the symbol 1, etc. The upper picture corresponds to two configurations (as shown, and reflected in the x-axis) and the lower corresponds to four (as shown, and reflected in the y-axis, the x-axis and the origin.)

when $n = 4$ equation (2.4) reduces to three coefficient equations: one linear, one quadratic and one cubic. Given generic values of the circulations, there are six equivalence classes of solutions $[u_1, u_2, u_3, u_4]$ (modulo multiplication of all u_j by a non-zero complex number). For each solution, if the u_j are distinct then they satisfy (2.2); if in addition they are all non-zero then they are cylinder coordinates for a stationary or translating point vortex street configuration. A typical solution set for a non-neutral stationary configuration of four streets is displayed schematically in figure 1. It can be seen in the upper part of the figure that the real parts of all the vortex positions are multiples of 1/2, corresponding to the fact that each u_j is real. Some streamlines of the fluid flow for these configurations are displayed in figure 2.

The coefficient equations obtained from equation (2.4) form a minimal polynomial system (O'Neil 2006) for stationary and translating configurations: Bezout's theorem implies that the total number of solutions to (2.4), when finite, is $(n-1)!$, and for topological reasons this is the number of stationary configurations when all circulations are positive (Montaldi, Soulière & Tokieda 2003). Thus there is no system of polynomials in the u_j of lower total degree with zero set corresponding to stationary or translating states. However for some circulation values the solution set is infinite, e.g. $\Gamma_1 = \Gamma_2 = 2$, $\Gamma_3 = \Gamma_4 = -1$, $[u_1, u_2, u_3, u_4] = [t, t^{-1}, 1, -1]$ where t is a free complex parameter. This example was mentioned in the introduction, and is a member of a family of stationary configurations of arbitrary complexity (Aref *et al.* 2003).

If one or more u_j is zero, say for $m < j \leqslant n$, then it is easy to show that (2.4) is also satisfied by $[u_1, \ldots, u_m, 0]$ where the last vortex has circulation $\Gamma_{m+1} + \cdots + \Gamma_n$. That is, all the vortices at $z = \mathrm{i}\infty$, $u = 0$, can be lumped together. Equation (2.2) shows that the effect of this street on the others is that of a uniform flow. Likewise, a translating configuration of vortex streets with real common velocity v can be converted to a solution of (2.2) with all $v_j = 0$ by adding a street with circulation $-2v$ at $u = 0$; note that the newly added street has zero velocity in (2.2) because the original translating configuration necessarily has zero total circulation.

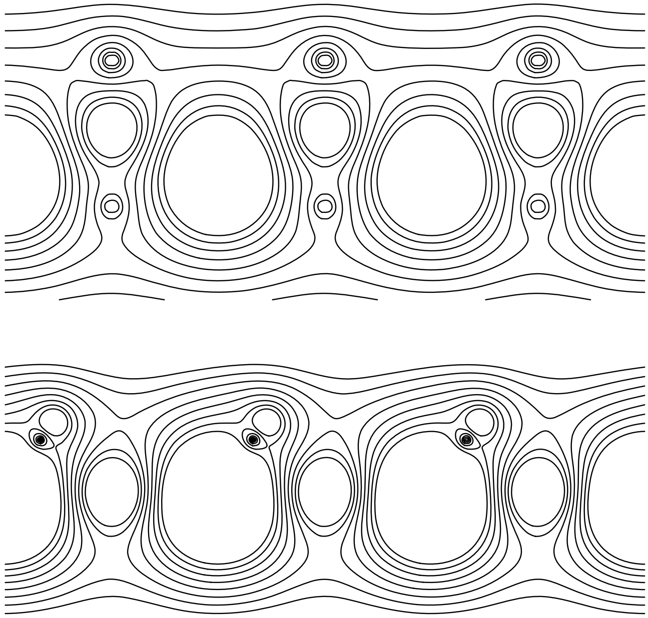

FIGURE 2. Streamlines for the two stationary configurations of figure 1. There are four stagnation points per period.

3. A differential equation for vortex street equilibria

Often a vortex street system will have several streets with the same circulation. Suppose the circulations take the distinct values Γ_α, $1 \leqslant \alpha \leqslant s$. Consider the s polynomials $P_\alpha(u) = \prod(u - u_j)$ where each product is taken over those indices j satisfying $\Gamma_j = \Gamma_\alpha$; thus the roots of P_α are the cylinder coordinates of all vortices with circulation Γ_α. The objective of this section is to show that equation (2.4) is satisfied if and only if these polynomials satisfy the differential equation (O'Neil 2006)

$$u\left(\sum_\alpha \Gamma_\alpha^2 \frac{P_\alpha''}{P_\alpha} + 2\sum_{\alpha<\beta} \Gamma_\alpha \Gamma_\beta \frac{P_\alpha' P_\beta'}{P_\alpha P_\beta}\right) + \sum_\alpha \Gamma_\alpha(\Gamma_\alpha - 2\bar{v} - S)\frac{P_\alpha'}{P_\alpha} = 0. \tag{3.1}$$

Multiply this equation by $P_1 \cdots P_s$ to see that it is linear in each P_α separately; in fact, it is a special case of Loutsenko's multilinear hypergeometric operator, which has been studied in the case $s = 2$ (Loutsenko 2003). Equation (3.1) is one of several differential equations that can be used to study relative equilibria of various vortex systems (Aref *et al.* 2003; Aref & Van Buren 2005; O'Neil 2006).

Because each P_α is a polynomial in u, so is the entire left-hand side of (3.1) after clearing denominators. Thus setting the coefficients equal to zero produces a system of polynomial equations in the coefficients of each P_α, a system that is sometimes easier to solve than the system derived from (2.4). The vortex street configurations can then be obtained by finding the roots of each of the P_α. Several examples of this process are given below. In addition, a simple calculation shows that the function $(2\pi i)^{-1}\log(Q(u)) - (S/2)z$, where $Q(u) = \prod P_\alpha{}^{\Gamma_\alpha}$, is a complex potential function for the flow in the complex plane due to the vortex street configuration. Thus the potential can be determined directly from the polynomial solutions to (3.1), without the need to compute the roots.

3.1. *Equivalence of the differential equation and equation* (2.4)

To obtain the differential equation from equation (2.4), first rewrite the double sum:

$$\frac{1}{2}\sum_{j\neq k}\Gamma_j\Gamma_k\frac{u_j+u_k}{(u-u_j)(u-u_k)}=\sum_j\left(\frac{\Gamma_j u_j}{u-u_j}\sum_{k:k\neq j}\frac{\Gamma_k}{u-u_k}\right)$$
$$=\left(\sum_j\frac{\Gamma_j u_j}{u-u_j}\right)\left(\sum_k\frac{\Gamma_k}{u-u_k}\right)+\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_j\frac{\Gamma_j^2 u_j}{u-u_j}\right). \quad (3.2)$$

Because of the identity

$$\sum_j\frac{\Gamma_j^p\, u_j}{u-u_j}=u\left(\sum_j\frac{\Gamma_j^p}{u-u_j}\right)-\sum_j\Gamma_j^p$$

for exponents $p=1,\ 2$, equation (2.4) is therefore equivalent to

$$\left(-2\bar{v}-S+u\sum_j\frac{\Gamma_j}{u-u_j}\right)\left(\sum_k\frac{\Gamma_k}{u-u_k}\right)+\frac{\mathrm{d}}{\mathrm{d}u}\left(u\sum_j\frac{\Gamma_j^2}{u-u_j}\right)=0. \quad (3.3)$$

In order to express this relation using only the polynomials $P_1,\ldots,P_s$, observe that

$$\sum_j\frac{\Gamma_j^p}{u-u_j}=\sum_\alpha\Gamma_\alpha^p\,\frac{P_\alpha'}{P_\alpha}$$

for $p=1,\ 2$, and

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u\sum_j\frac{\Gamma_j^2}{u-u_j}\right)=\sum_\alpha\Gamma_\alpha^2\,\frac{P_\alpha'}{P_\alpha}+u\sum_\alpha\Gamma_\alpha^2\,\frac{P_\alpha''}{P_\alpha}-u\sum_\alpha\left(\frac{\Gamma_\alpha P_\alpha'}{P_\alpha}\right)^2.$$

Substitution of these identities into (3.3) yields (3.1).

In summary, if $P_1,\ldots,P_s$ satisfy (3.1) with $S=\sum_\alpha\Gamma_\alpha\deg(P_\alpha)$ and the roots of these polynomials are all distinct and non-zero, then these roots are the cylinder coordinates of a point vortex street configuration that is translating with velocity v (in the neutral case) or is stationary (non-neutral case.)

The invariance of (2.1) and (2.2) under reflections noted in § 2 extends to (3.1) in the following way. Suppose that $P_1,\ldots,P_s$ satisfy (3.1) with no roots at zero. Since the mapping $u_j\mapsto 1/u_j$ takes each polynomial P_α to a non-zero constant times its reversal

$$R(P_\alpha):=u^{\deg P_\alpha}\ P_\alpha(1/u),$$

the invariance of (2.2) under $u_j\mapsto 1/u_j$, $v_j\mapsto -v_j$ shows that (3.1) is invariant under the involution $P_j\mapsto R(P_j)$, $v\mapsto -v$.

3.2. *Solving the differential equation for small s*

The left-hand side of the differential equation (3.1), after clearing denominators, is clearly a polynomial in u of degree no greater than $n-1$; in fact, the coefficient of u^{n-1} is zero so that it is actually of degree $n-2$. Thus its vanishing is equivalent to a system of $n-1$ nonlinear algebraic equations in the coefficients of the unknown polynomials P_α. Each equation has degree s and is linear in each variable separately. When the number of polynomials (or the degrees of some of the polynomials) is large, the system is intractable. On the other hand, less complex cases can sometimes be solved exactly with very little difficulty.

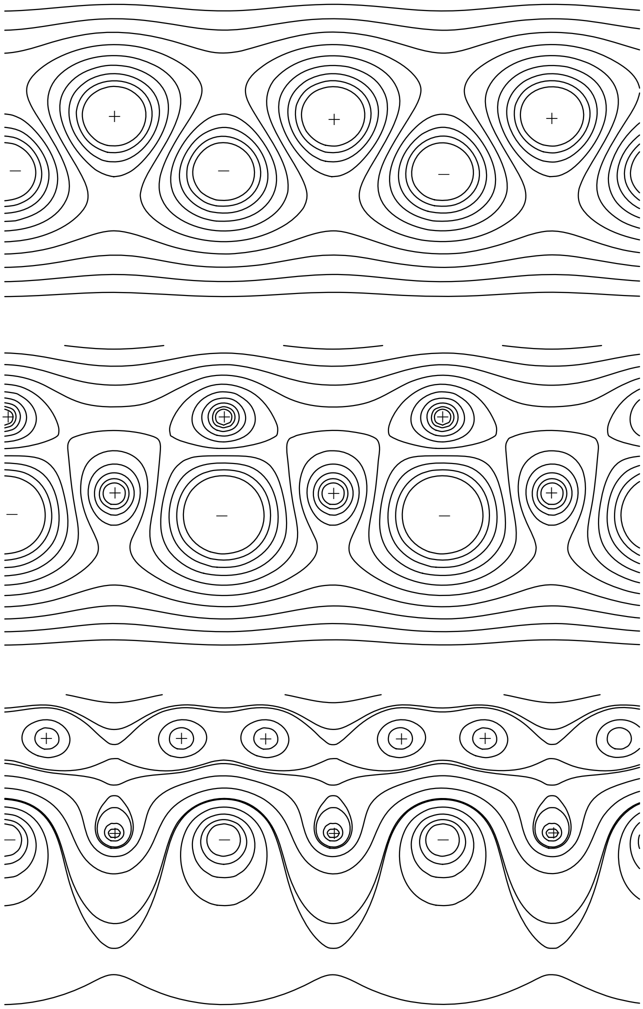

Figure 3. Streamlines of several translating neutral configurations as they appear in a frame moving with the vortices. All the positive vortices have circulation 1; from the top, Γ_1 has the value -1, -2 and -3.

Consider a system of vortex streets where the first has circulation Γ_1 and all the rest have circulation Γ_2, so that $s=2$ and $\deg(P_1)=1$. Multiplying the u_j by an appropriate non-zero constant, we may assume $u_1=1$ so that $P_1=u-1$. The differential equation (3.1) then becomes a *linear* system for the $n-1$ coefficients of P_2; an explicit formula for this solution polynomial is given in equation (6.2) below. If the street system is neutral, e.g. $\Gamma_1=1-n$ and $\Gamma_2=1$, one obtains a translating configuration in which the velocity is a parameter. These configurations can be used as a starting point for the construction described in § 5. The simplest configuration, with $n=2$, is the elementary alternating-vortex model of the wake of a bluff body; setting the common velocity v to the value 0.354 gives the Kármán wake. Larger values of n produce configurations where the positive vorticity is equally distributed among more numerous, weaker vortices. Figure 3 shows the streamlines of the fluid flow induced by several such configurations, as they appear in a frame in which the vortices are

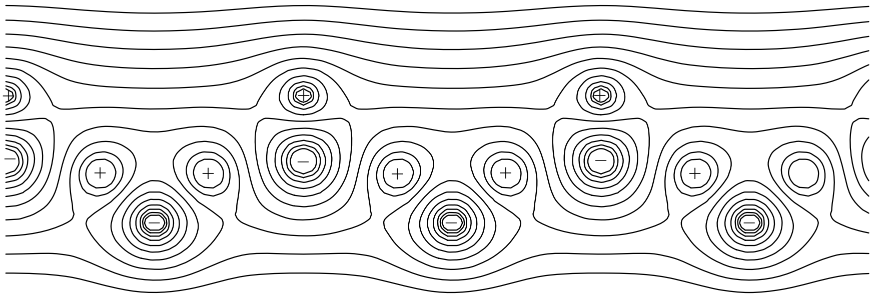

FIGURE 4. Streamlines of a translating neutral configuration as viewed in a frame moving with the vortices. All positive vortices have circulation 1, and all negative ones have circulation −3/2.

at rest. In this frame the complex potential for the flow is

$$(2\pi \mathrm{i})^{-1} \log(Q(u)) - vz = (2\pi \mathrm{i})^{-1} \log(Q(u)\, u^{-v}),$$

so that the streamlines are the level sets of $\ln |Q(u)\, u^{-v}|$. In order to keep a constant ratio of velocity to vorticity per period, the velocity parameter used for each configuration pictured is $v = 0.354(n-1)$. The n stagnation points in each plot are clearly visible.

It is only a little harder to solve (3.1) when $s = 2$ and there are two streets with circulation Γ_1, i.e. the polynomial P_1 is quadratic. We may take $P_1 = (u-1)(u-x)$ so that the differential equation reduces to a system of quadratic equations, linear in x and also linear in the unknown coefficients of P_2. The system can be reduced to a single polynomial equation in x alone, and the value of x determines P_2. A neutral translating configuration with two negative and three positive vortices per period found in this way is shown in figure 4. The normalized velocity is the same as in the previous figure.

Now consider the case $s = 3$. Suppose there are $n-2$ vortices of circulation Γ_1 and one each of circulations Γ_2 and Γ_3 per period. We may put $P_2 = u - 1$, $P_3 = u - x$ and solve for unknowns x and $P_1 = u^{n-2} + a_{n-3}u^{n-3} + \cdots + a_0$. Again the system of equations is in effect a collection of recursion relations for the coefficients a_j. Specifically, the equation corresponding to the highest power of u in (3.1) involves only x and a_{n-3}, so that a_{n-3} may be written as a linear polynomial in x; the next equation may be solved for a_{n-4} as a function of x and a_{n-3}, so that a_{n-4} is a quadratic function of x, and so on. Thus all the equations but one may be viewed as determining P_1 from x. The final equation (i.e. the one obtained from (3.1) by setting $u = 0$) then reduces to a polynomial of degree $n-1$ in x alone. Each of the $n-1$ solutions for x then determines a corresponding polynomial P_1. Since P_1 has $n-2$ roots corresponding to identical vortices, one obtains by permutation the complete set of $(n-1)!$ solutions to equation (2.4).

For example, take $n = 5$, $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, -4/3, -5/3)$ and $2\bar{v} = 1/2$ in (3.1). A minimal rearrangement of the coefficient equations yields the system

$$\begin{aligned} a_2 &= (20 + 35x)/27, \\ a_1 &= (-a_2 - x(135 - 2a_2))/90, \\ a_0 &= (14a_1 + x(5a_1 - 54a_2))/189, \\ 0 &= 11\,x^4 - 5956\,x^3 - 8934\,x^2 - 6868\,x - 13. \end{aligned}$$

Changing circulations to $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, -1/2, -5/2)$ and $2\bar{v}$ to 5 produces the system:

$$\begin{aligned}
a_2 &= -(9 + 25\,x)/24,\\
a_1 &= (11a_2 + x(24 - 21a_2))/56,\\
a_0 &= (13a_1 + x(-3a_1 + 8a_2))/32,\\
0 &= (99\,x^2 - 58\,x + 99)(35\,x^2 - 154\,x + 195).
\end{aligned}$$

Observe that the quartic polynomial for x factors into two quadratics; this simplifies the final expressions for x and P_1. This example will be continued in § 5.3.

4. Loutsenko's stationary configurations and related translating configurations

If there are only two values of circulations, $\Gamma_2/\Gamma_1 = \Gamma$, then with the simplified notation $P_1 = p$, $P_2 = q$ and $(2\bar{v} + S)/\Gamma_1 = \chi$, equation (3.1) takes the form

$$u(p''q + \Gamma^2 p\,q'' + 2\Gamma p'q') + (1 - \chi)p'q + \Gamma(\Gamma - \chi)p\,q' = 0. \tag{4.1}$$

Loutsenko constructed trigonometric polynomials with roots corresponding to non-neutral stationary ($S \neq 0$, $v = 0$) configurations of vortex streets where all vortex circulations have the same absolute value, i.e. $\Gamma = -1$ (Loutsenko 2003). The polynomials contain complex parameters and so establish a continuum of configurations. It will now be shown that each such configuration is connected via a certain limit to a neutral translating configuration, and that the same construction can produce translating configurations with arbitrary velocity.

Let $a_0, \ldots, a_n$ be distinct positive integers and $c_0, \ldots, c_n$ be complex numbers. Define the functions $f(z)$ and $g(z)$ to be the Wronskian determinants

$$\begin{aligned}
f(z) &= \det\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\sin\pi(a_j z + c_j)\right]_{0\leqslant j,\,k\leqslant n-1},\\
g(z) &= \det\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\sin\pi(a_j z + c_j)\right]_{0\leqslant j,\,k\leqslant n}.
\end{aligned}$$

Then the zeros of f and g correspond to the location of negative and positive unit circulation vortices respectively in a stationary configuration of point vortex streets. These functions may be expressed in terms of the cylinder coordinates $u_j = \exp(2\pi i z_j)$. Writing $d_j = \exp(2\pi i c_j/a_j) \neq 0$, yields

$$2i\frac{\mathrm{d}^k}{\mathrm{d}z^k}\sin\pi(a_j z + c_j) = (d_j u)^{-a_j/2}((d_j u)^{a_j} - (-1)^k)(i\pi a_j)^k.$$

After some manipulation it is found that the cylinder coordinates of the zeros of $f(z)$ and $g(z)$ are exactly the zeros of polynomials $q(u)$ and $p(u)$ respectively:

$$q(u) = \det\left[((d_j u)^{a_j} - (-1)^k)a_j^k\right]_{0\leqslant j,\,k\leqslant n-1}, \tag{4.2}$$

$$p(u) = \det\left[((d_j u)^{a_j} - (-1)^k)a_j^k\right]_{0\leqslant j,\,k\leqslant n}. \tag{4.3}$$

Since all d_j are non-zero, $\deg(q) = a_0 + \cdots + a_{n-1}$ and $S = \deg(p) - \deg(q) = a_n$. Hence p, q satisfy (4.1) with $\Gamma = -1$, $\chi = a_n$. Now (4.1) is satisfied for all values of the constants c_j, so take the limit $c_n \to i\infty$, that is, $d_n \to 0$. In this limit the polynomial p has degree $a_0 + \cdots + a_{n-1}$ (a_n of the roots having diverged to infinity) so that p, q now satisfy (4.1) with the same parameters $\Gamma = -1$, $\chi = a_n$ but now $2\bar{v} = \chi$

and $S=0$. Physically, a_n of the positive vortex streets have moved to $-\mathrm{i}\infty$ leaving the remaining (neutral) system at rest in the uniform flow $-a_n\Gamma_1/2$. Removing this uniform background reveals a translating neutral system.

The formula (4.3) for p allows us to say a little more about the residual translating system. The integer parameter a_n no longer appears in any exponent of any term of p, so that the left-hand side of equation (4.1) is a polynomial in a_n. Since this polynomial is known to have the value zero for all sufficiently large integers a_n, it must be the zero polynomial. Thus (4.1) continues to hold for any *complex* value for a_n. In other words, p and q as defined in (4.2), (4.3) with $d_n=0$ and a_n an arbitrary complex number are polynomial solutions of (4.1) with $\Gamma=-1$, $S=0$ and $2\bar{v}=a_n\Gamma_1$ so long as $a_0,\ldots,a_{n-1}$ are distinct positive integers.

This leaves us with the following picture: given n distinct positive integers $a_0,\ldots,a_{n-1}$, there is a corresponding solution set of translating point vortex street configurations with n complex parameters, one parameter being the translation velocity; certain of these configurations are limits of families of stationary street configurations with one additional parameter.

When n is small the determinants in (4.2), (4.3) can be computed quickly. The simplest polynomial pair is obtained when $n=1$: combining and normalizing some constants, we have

$$q=u^{a_0}-1,$$
$$p=(a_0-a_1)+(a_0+a_1)u^{a_0}+tu^{a_1}((a_0-a_1)u^{a_0}+(a_0+a_1)),$$

t being the single complex parameter. The a_0 negative vortices in each period are evenly spaced along a line in the x-direction, while the positions of the a_0+a_1 positive vortices depend on the parameter t. As t goes to zero, a_1 of the positive vortex streets move off to infinity leaving a neutral configuration behind. The symmetry present in the polynomial p clearly comes from the determinant (4.3), but a more general explanation appears in § 6.

5. Connected stationary and translating configurations for more general circulations

In this section we establish that the pattern just observed for positive and negative unit strength vortices can also be found in other vortex street systems that have two or more circulation values. The primary restrictions are that all circulations have the same sign save for one, and are half-integral multiples of the negative of this exceptional circulation. The basic technique, reduction of order, has been used to find stationary vortex configurations in the infinite plane (Loutsenko 2004) and is described in § 5.1. The integral relation is recast as a more convenient differential relation in § 5.2, yielding a streamlined algorithm and the degree relations that validate the claim that began this paragraph. Some examples illustrating the general case are given in § 5.3.

5.1. *Construction of a continuum of solutions to* (3.1)

Suppose $P_1,\ldots,P_s$ satisfy the differential equation (3.1). After multiplication by P_1, this equation can be viewed as a linear second-order differential equation in P_1 (holding the other polynomials constant), so that the method of reduction of order can be used to find the second independent solution $\tilde{P}_1$. Recall the definitions $Q(u)=\prod P_\alpha^{\Gamma_\alpha}=\prod(u-u_k)^{\Gamma_k}$ and $\chi=(2\bar{v}+S)/\Gamma_1$. A straightforward calculation shows that the functions $\tilde{P}_1, P_2, \ldots, P_s$ also satisfy (3.1), where

$$\tilde{P}_1 = P_1 \int Q^{-2/\Gamma_1} u^{\chi-1}\,\mathrm{d}u. \tag{5.1}$$

We shall now see that under certain circumstances $\tilde{P}_1$ is a polynomial, so that for arbitrary constants C_1 and C_2 the polynomials $(C_1 P_1 + C_2 \tilde{P}_1), P_2, \ldots, P_s$ satisfy (3.1), and the roots of these polynomials form a parametrized family of vortex street configurations that will be proved below to be connecting stationary and translating systems. The roots of the first polynomial are determined by the ratio C_2/C_1, so the solution set has one complex dimension.

For simplicity, assume that (i) the roots of P_1 are all non-zero, and are not roots of the other polynomials $P_2, \ldots, P_s$. It is clear that $\tilde{P}_1$ can only be a polynomial if the integrand of (5.1) is a rational function with no poles other than at the zeros of P_1. Therefore assume further that (ii) each $-2\Gamma_\alpha/\Gamma_1$, $1<\alpha$ is a positive integer and (iii) χ is a positive integer. The roots of P_1 are necessarily distinct, because otherwise the left-hand side of (2.4) would have a double pole at the repeated root, and hence could not be zero. Thus under assumptions (i)–(iii) the integrand of (5.1) is a rational function with a pole of order two at each root of P_1. Let u_j be a root of P_1 and $Q_j(u) = \prod_{k:k\neq j}(u-u_k)^{\Gamma_k}$. The residue of the integrand at $u=u_j$ is zero exactly when the relation

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=u_j} \left(Q_j^{-2/\Gamma_1} u^{\chi-1}\right)$$

holds, or equivalently,

$$(2/\Gamma_1) Q_j'(u_j) u_j = Q_j(u_j)(\chi - 1). \tag{5.2}$$

To see that (5.2) does indeed hold, substitute $v_j = v$ into (2.2):

$$2\bar{v} = \sum_{k:k\neq j} \Gamma_k \frac{u_j + u_k}{u_j - u_k} = \sum_{k:k\neq j} \Gamma_k \left(-1 + 2u_j \frac{1}{u_j - u_k}\right) = \Gamma_1 - S + 2u_j \frac{Q_j'(u_j)}{Q_j(u_j)}$$

and recall that $2\bar{v} + S - \Gamma_1 = \Gamma_1(\chi - 1)$. Thus the residues of these poles are all zero, the only singularities of the integral are simple poles – that is, there are no logarithmic terms on the right-hand side of (5.1) – and $\tilde{P}_1$ is a polynomial.

5.2. *A first-order differential relation for* $\tilde{P}_1$

Equation (5.1) does not represent the most efficient method for computing $\tilde{P}_1$. Divide (5.1) by P_1, differentiate and multiply by uP_1^2 to obtain the relation

$$u\left(\tilde{P}_1' P_1 - \tilde{P}_1 P_1'\right) = u^\chi \prod_{1<\alpha} P_\alpha^{-2\Gamma_\alpha/\Gamma_1}. \tag{5.3}$$

This shows that the polynomial $\tilde{P}_1$ may be found from $P_1, \ldots, P_s$ by solving a system of linear equations in its coefficients. Of course $\tilde{P}_1 = P_1$ is a solution to the homogeneous equation $u(\tilde{P}_1' P_1 - \tilde{P}_1 P_1') = 0$, corresponding to the term resulting from the constant of integration in equation (5.1). Suppose a_j, $\tilde{a}_j$ and c_j are the coefficients of u^j in P_1, $\tilde{P}_1$ and the polynomial $\prod_{1<\alpha} P_\alpha^{-2\Gamma_\alpha/\Gamma_1}$ respectively, the $\tilde{a}_j$ being unknown. Then comparing like powers of u on the two sides of (5.3) produces a sequence of linear equations:

$$\sum_{j+k=m} \tilde{a}_j\, a_k (j-k) = c_{m-\chi}, \quad m = 0, 1, \ldots \tag{5.4}$$

The solution can be found by setting $\tilde{a}_j = 0$ for $j < \chi$ and then using (5.4) to compute sequentially the coefficients $\tilde{a}_\chi$, $\tilde{a}_{\chi+1}$, etc. Thus the full solution $C_1 P_1 + C_2 \tilde{P}_1$ to (5.3) will have the form $C_1 P_1 + C_2 u^\chi h(u)$ for some polynomial $h(u) = c_0/(\chi a_0) + \cdots$.

Equation (5.3) implies that the degree of $\tilde{P}_1$ satisfies the relation

$$\deg P_1 + \deg \tilde{P}_1 = \chi - (2/\Gamma_1) \sum_{1<\alpha} \Gamma_\alpha \deg P_\alpha. \tag{5.5}$$

Now $S = \sum_\alpha \Gamma_\alpha \deg P_\alpha$ is the total circulation of the system corresponding to the roots of $P_1, \ldots, P_s$. Let $\tilde{S}$ denote the total circulation of the new system that substitutes $\tilde{P}_1$ for P_1; then (5.5) reduces to $S + \tilde{S} = \Gamma_1 \chi$, i.e. $\tilde{S} = 2\bar{v}$. If $S \neq 0$ then the initial configuration is stationary and the new configuration is neutral, $\tilde{S} = 0$, and contains χ fewer vortices of circulation Γ_1 per period; whereas if the original configuration is neutral then the new configuration will be stationary and contain $\chi = 2\bar{v}/\Gamma_1$ additional vortices of circulation Γ_1. We conclude therefore that the vortex street configurations associated with the polynomials $(C_1 P_1 + C_2 \tilde{P}_1)$, $P_2, \ldots, P_s$ will be stationary for non-zero C_1, C_2, and tend to a translating configuration as the constant multiplying the polynomial of larger degree goes to zero. Thus the one-dimensional set of solution polynomials connects stationary and translating configurations.

A remark may be made in passing about the differential relation (5.3) used to compute $\tilde{P}_1$. Analogous relations generate the sequence of polynomials describing non-periodic stationary configurations of unit vortices (Burchnall & Chaundy 1930; Adler & Moser 1978), and another sequence of polynomials (Loutsenko 2004) for stationary configurations when the circulation ratio is -2. The idea is to interchange the roles of P_1 and P_2 after each application of (5.1), with the constants of integration becoming free parameters in the polynomials. Unfortunately, in the present circumstance the factor u^χ on the right-hand side of (5.3) is an obstacle to the creation of a similar sequence of polynomials for vortex streets, since χ would change sign with each step. Fortunately, that same factor allows one to create an infinite number of one-dimensional families of stationary configurations by having the initial translating configuration move at different velocities (different integral values of χ.)

5.3. *Examples*

The construction described above is at its most flexible when starting with a neutral system. By choosing different translation velocities, one can create many one-dimensional families of stationary configurations.

As a simple example, we may begin with a translating configuration consisting of streets with circulations ± 1. The case of one negative street will be covered by the explicit formula (6.2) in the next section, so begin instead with two positive and two negative streets. The determinant formulae (4.2), (4.3) allow us to quickly find polynomials $P_1 = (2+\chi)u^2 + (2-\chi)$ and $P_2 = u^2 - 1$ determining the positive and negative streets respectively. To find a family of stationary configurations with one additional positive street, set $\chi = 1$ and solve the system (5.4) to obtain $\tilde{P}_1 = u^3/3 + u$, so that the positive streets are determined by the roots of $C_1(3u^2+1) + C_2 u(u^2/3+1)$. It is just as easy to add m positive streets by following the same procedure with $\chi = m$, so long as $m \neq 2$. A symmetry may be observed between P_1 and $\tilde{P}_1$ in all these cases; this is discussed in § 6.

For an example involving more than two circulations that is not overly complicated, consider the neutral system of three vortex streets of circulation $\Gamma_1 = 1$ and one each of circulations $\Gamma_2 = -1/2$ and $\Gamma_3 = -5/2$; this system was considered in § 3. Suppose one wishes to construct the continuum of stationary configurations with total vorticity 5

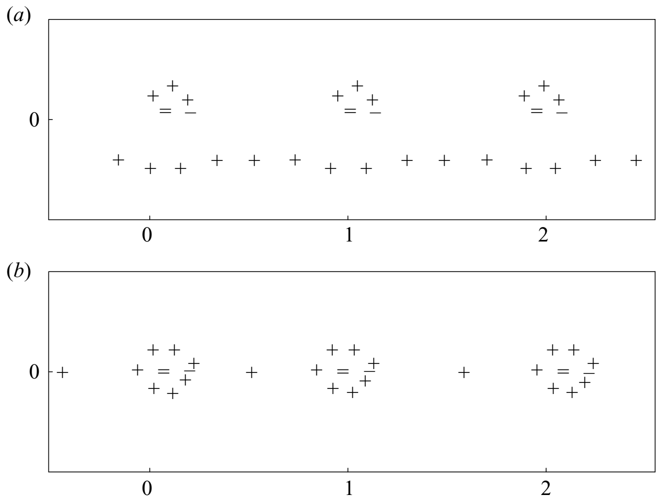

FIGURE 5. Stationary vortex street configurations for the polynomials (5.6), (5.7). The symbols $+$, $-$, $=$ represent vortex streets with circulations 1, $-1/2$, $-5/2$ respectively. Frame ticks at integer x and y values emphasize the periodicity. The value of the parameter t is 1 in (a) and $(1+\mathrm{i})/300$ in (b).

that connect with this translating system. Equation (3.1) with $\chi = 2\bar{v} = 5$ is satisfied by the polynomials

$$\left.\begin{aligned} P_1(u) &= 231u^3 - (266 + 14\mathrm{i}\sqrt{35})u^2 + (129 + 12\mathrm{i}\sqrt{35})u - (24 + 3\mathrm{i}\sqrt{35}), \\ P_2(u) &= u - (29 + 16\mathrm{i}\sqrt{35})/99, \quad P_3(u) = u - 1. \end{aligned}\right\} \tag{5.6}$$

The polynomial $\tilde{P}_1$ will have degree 8; application of (5.4) yields (up to multiplicative constant)

$$27u^8 - (132 + 3\mathrm{i}\sqrt{35})u^7 + (238 + 14\mathrm{i}\sqrt{35})u^6 - (168 + 21\mathrm{i}\sqrt{35})u^5 + tP_1(u) \tag{5.7}$$

with free complex parameter t. In the limit $t \to \infty$ five streets move off to $-\mathrm{i}\infty$, leaving a neutral translating configuration. This translating state is taking shape in figure 5(a). This configuration has the appearance of a neutral intermittent wake interacting weakly with a simple shear layer, with some rather intricate streamlines as seen in figure 6. The five streets move in the opposite direction as $t \to 0$, leaving a translating configuration that is the reflection in the y-axis. Several parameter values cause positive vortices to coalesce on one of the negative vortices, as suggested by figure 5(b) where the weaker negative vortices combine with positive ones to form stationary 'tripoles'.

To add six rather than five vortices to the system, merely go back to (3.1) and use $\chi = 6$ to calculate the starting configuration

$$P_1(u) = 546u^3 - (792 + 18\mathrm{i}\sqrt{105})u^2 + (441 + 18\mathrm{i}\sqrt{105})u - (90 + 5\mathrm{i}\sqrt{105}),$$
$$P_2(u) = u - (73 + 12\mathrm{i}\sqrt{89})/255, \quad P_3(u) = u - 1$$

and again apply (5.4).

By relaxing the simplifying assumption (i) used above, the procedure can also be used on translating configurations that have some streets at $u = 0$, although the zero-residue criterion (5.2) is not valid in this case. For instance, suppose $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, -1/2, -3/2)$ and the initial translating configuration corresponds

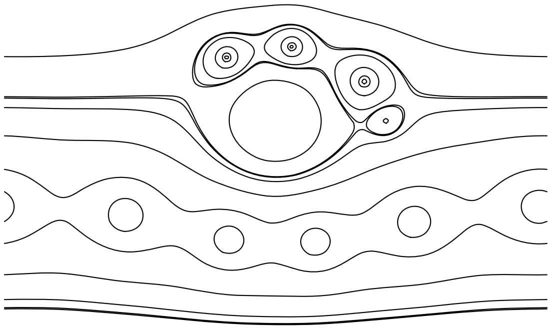

FIGURE 6. Some streamlines for the configuration in figure 5(a).

to $Q = u^2(u-1)^{-1/2}(u+1)^{-3/2}$ and $\chi = 2$. An application of (5.1) yields the new polynomial $1 + 4u + t\,u^2 + 4u^3 + u^4$ with complex parameter t. As $t \to \infty$, half the streets of circulation 1 move to $\mathrm{i}\infty$ and the other half move to $-\mathrm{i}\infty$. Many similar solution families (including the one mentioned in the introduction) can be generated with ease in this way.

6. Partially symmetric systems

The full solution $C_1 P_1 + C_2 u^\chi h(u)$ to (5.3) has a particularly simple form if $S = 0$ and the zeros of the other polynomials are symmetrically distributed. Suppose that the polynomial P_j has the same roots as the reversed polynomial $R(P_j)$ for each $j > 1$; this means that point vortices of circulation Γ_j are symmetric with respect to reflection through the origin. Suppose further that $P_1, \ldots, P_s$ satisfy (3.1) with $S = 0$ and $2\bar{v} = m\Gamma_1$ for some positive integer m. Using the invariance of (3.1) under the involution $P_\alpha \mapsto R(P_\alpha)$, $v \mapsto -v$ we find that $R(P_1), P_2, \ldots, P_s$ satisfy (3.1) with $S = 0$ and $2\bar{v} = -m\Gamma_1$. As observed at the end of § 2, the configuration can be converted to a stationary one by adding m vortices of circulation Γ_1 at $u = 0$, so that $u^m R(P_1), P_2, \ldots, P_s$ satisfy (3.1) with $S = m\Gamma_1$ and $2\bar{v} = 0$. Since the parameters v and S only appear in (3.1) as the sum $2\bar{v} + S$, it is clear that $P_1, P_2, \ldots, P_s$ and $u^m R(P_1), P_2, \ldots, P_s$ satisfy the same differential equation. In other words, $h(u) = R(P_1)$.

The simplest example has $q = (1-u)$ so that (4.1) becomes Gauss's hypergeometric differential equation,

$$u(1-u)p'' + \{(1-\chi) - u(\Gamma + (\Gamma - \chi) + 1)\}p' - \Gamma(\Gamma - \chi)p = 0. \tag{6.1}$$

Assuming a total of $n > 2$ streets, if the system is not neutral then $\Gamma - \chi = 1 - n = -\deg p$, whereas for neutral systems $\Gamma = 1 - n$ and $\Gamma - \chi = (\Gamma_2 - 2\bar{v})/\Gamma_1$. One solution to (6.1) is the hypergeometric series

$$p(u) = 1 + \frac{\Gamma(\Gamma-\chi)}{1-\chi}\,u + \frac{\Gamma(\Gamma+1)(\Gamma-\chi)(\Gamma-\chi+1)}{2!(1-\chi)(2-\chi)}\,u^2 + \cdots \tag{6.2}$$

For a neutral system ($\Gamma = 1 - n$) the series terminates, i.e. $p(u)$ is a polynomial, so long as $\chi = 2\bar{v}/\Gamma_1$ is not one of the integers $1, \ldots, n-1$. Hence the method of the preceding section can be used to generate families of stationary configurations containing n or more additional vortices of strength Γ_1. Since q and $R(q)$ have the same root, it follows from the discussion above that $\tilde{p} = u^\chi R(p)$. For example, starting with a neutral system of three vortices of strength 1 and one of strength $\Gamma = -3$, putting $\chi = 5$ in (6.2) gives the hypergeometric polynomial $1 - 6u + 14u^2 - 14u^3$,

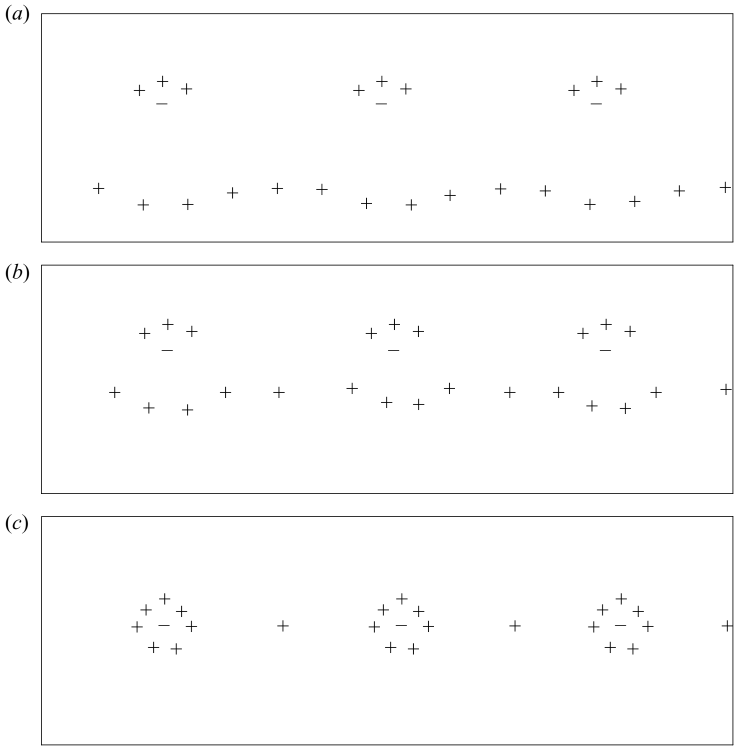

FIGURE 7. Stationary vortex street configurations corresponding to (6.3). The value of the parameter C_1/C_2 is 1000 (*a*), 10 (*b*) and 1.005 (*c*).

leading directly to a one-parameter solution to (4.1):

$$q = u - 1, \;\; p = C_1(1 - 6u + 14u^2 - 14u^3) + C_2u^5(u^3 - 6u^2 + 14u - 14). \qquad (6.3)$$

Some representative configurations are shown in figure 7. Figure 7(*a*) can be interpreted as a translating wake-like configuration interacting weakly with a shear layer, and the other figures show the result of bringing these entities into close contact.

There is another symmetry of equation (3.1), related to decreasing the period of the street configuration, that allows other solution families to be described easily. Suppose that $P_1(u), \ldots, P_s(u)$ is a solution to (3.1) and k is a positive integer. It is not difficult to show that (3.1) and (4.1) continue to be satisfied after the substitutions $P_\alpha(u) \mapsto P_\alpha(u^k)$, $v \mapsto kv$ and $\chi \mapsto k\chi$. (It is implicit that the total circulation after substitution is k times its previous value.) The number of added streets in the stationary configurations need not be a multiple of k, however, leading to interesting accommodations between the fixed streets and the parameter-dependent streets. For example, starting with a neutral system of three vortices of strength 1 and one of strength -3, making the substitution $u \mapsto u^3$ and then putting $\chi = 5$ leads to the solution polynomials $q = u^3 - 1$ and

$$p = C_1(-1 + 21u^3 + 231u^6 + 154u^9) + C_2u^5(-u^9 + 21u^6 + 231u^3 + 154). \qquad (6.4)$$

Some street configurations are shown in figure 8.

Another interesting class of examples involves an 'alternating street'. For a three-species system ($s = 3$) with $\Gamma_1 = 1$, let deg $P_1 = -(\Gamma_2 + \Gamma_3)$ and set $P_2 = u - 1$, $P_3 = u + 1$ so that the two polynomials combine to create a street with vortex spacing 1/2 and vortex circulations that are alternately Γ_2 and Γ_3. With the assistance of a computer

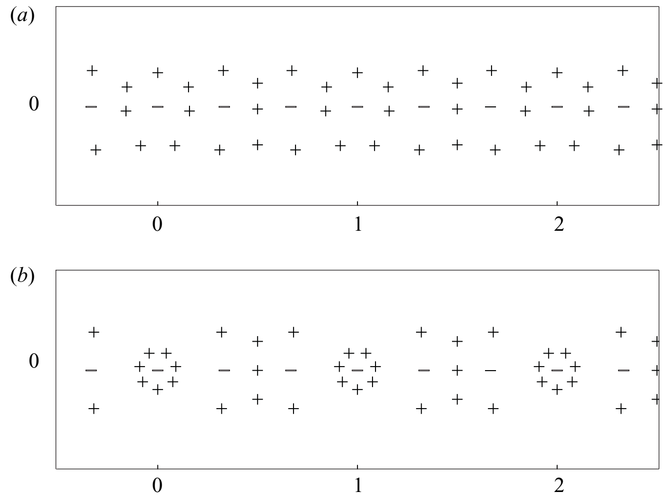

FIGURE 8. Stationary vortex street configurations corresponding to polynomials (6.4). The value of the parameter C_2/C_1 is 0.25 (*a*) and 1.01 (*b*).

algebra system, one can show that the linear system of equations determining the coefficients of a polynomial P_1 satisfying (3.1) has a non-zero solution only when Γ_2 and Γ_3 are negative integers. For example, circulations −1, −2 yield the polynomial

$$P_1(u) = 1 + \left(\frac{\chi+3}{\chi-1}\right)\left(-u - u^2 + \frac{\chi+1}{\chi-3}u^3\right).$$

The method of the previous section can now be used to create stationary configurations by adding any number of positive vortices except for 1 or 3. Adding 2, for example, and observing $R(P_2) = -P_2$, $R(P_3) = P_3$, yields the polynomial

$$C_1(1 - 5u - 5u^2 - 15u^3) + C_2u^2(u^3 - 5u^2 - 5u - 15). \tag{6.5}$$

Figure 9 shows representative stationary configurations.

7. Translating and stationary configurations with $\Gamma = -2$

There is no known closed form for solutions to (4.1) in the case $\Gamma = -2$ of the sort discussed in § 4, but some polynomial solutions may be found by direct calculation. In addition to isolated solutions there are also solutions with a free parameter that are analogous to polynomial solutions in the $\Gamma = -1$ case. These are found by picking integers $0 < m < n$ and giving q the special form

$$q(u) = u^n + u^{n-m} + tu^m + t\left(\frac{(n-2m)}{n}\right)^2 \frac{(2n-3m)^2 + 3(m^2 - \chi^2)}{(2n-m)^2 + 3(m^2 - \chi^2)}$$

in (4.1), which then becomes a recursion relation for the coefficients of p. The single free parameter t appears in both p and q, as does the translation parameter χ (in contrast to the configurations of § 4, in which q did not vary with χ at all.) A typical example is

$$q(u) = u^5 + u^3 + tu^2 + t\frac{(3\chi^2 - 28)}{25(3\chi^2 - 76)},$$

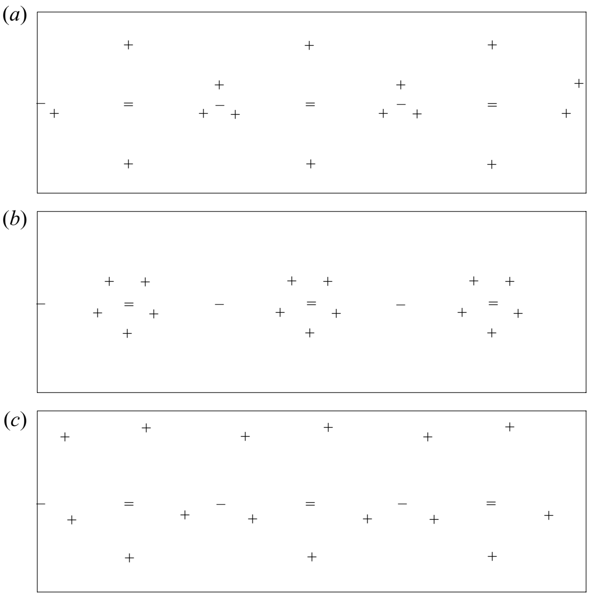

FIGURE 9. Stationary vortex street configurations for the 'alternating street' example (6.5). The symbols −, = and + indicate vortices of circulation −1, −2 and 1 respectively. The three values of the parameter ratio C_2/C_1 are 1.2 (*a*), −1.04 (*b*) and −10 i (*c*).

$$\begin{aligned}
p(u) = {} & u^{10} + 2\frac{(\chi-4)}{(\chi+2)}\,u^8 + 2t\frac{(\chi-6)}{(\chi+3)}\,u^7 + \frac{(\chi-2)(\chi-4)}{(\chi+2)(\chi+4)}\,u^6 \\
& + 4t\frac{(\chi-6)(\chi-4)(39\chi^2-316)}{25(\chi+2)(\chi+3)(3\chi^2-76)}\,u^5 + t^2\frac{(\chi-3)(\chi-6)}{(\chi+3)(\chi+6)}\,u^4 \\
& + 2t\frac{(\chi-2)(\chi-4)(\chi-6)(3\chi^2-28)}{25(\chi+2)(\chi+3)(\chi+4)(3\chi^2-76)}\,u^3 \\
& + 2t^2\frac{(\chi-3)(\chi-4)(\chi-6)(3\chi^2-28)}{25(\chi+2)(\chi+3)(\chi+6)(3\chi^2-76)}\,u^2 \\
& + (t/25)^2\frac{(\chi-2)(\chi-3)(\chi-4)(\chi-6)(3\chi^2-28)^2}{(\chi+2)(\chi+3)(\chi+4)(\chi+6)(3\chi^2-76)^2}.
\end{aligned}$$

The translating configurations make up a one-dimensional set for any integral χ other than −2, −3, −4 or −6. Figure 10 shows several configurations based on this translating configuration, with five vortices of circulation 1 added. Because the translating configuration has a free parameter, the result is a two- (complex) dimensional set of stationary configurations. One parameter affects all vortices while the other affects the positions of the positive vortices only. As with previous examples, there are values of the parameters that cause p, q to have roots in common. Similarly one can construct other families by adding k streets of circulation −2 as long as $k \neq 1$, 2 or 3.

8. Summary

In this paper, a class of stationary periodic point vortex configurations having a free complex parameter was proved to exist, subject to some restrictions on the vortex circulations. The basic tool was a differential equation for the polynomials describing the configurations; this differential equation is equivalent to a system of polynomial

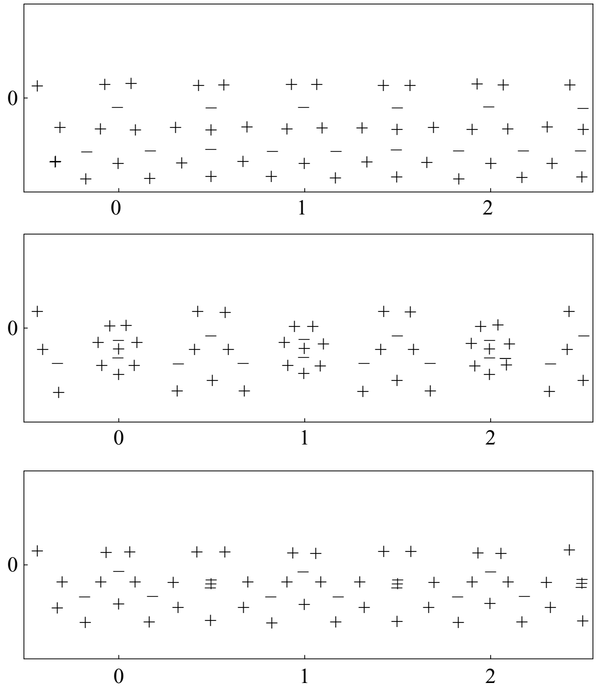

FIGURE 10. Several stationary vortex street configurations with fifteen vortices of circulation 1 and five of circulation −2.

equations in the coefficients of the polynomials. For one value of the parameter, the configurations take on a limiting form that is a neutral translating configuration. Conversely, given such a translating configuration, there is an effective procedure for determining the stationary configurations that involves solution of a linear system of equations. Several methods for finding translating configurations were presented. One can solve the differential equation directly to find simple configurations. Alternatively, one can take advantage of explicit forms for systems with circulations 1 and −1 (§ 4), or circulations 1 and −2 (§ 7), or use a hypergeometric function (§ 6). In addition, some formal properties of the polynomials related to symmetries can be exploited to further simplify the calculation.

All these configurations represent singular solutions of the two-dimensional Euler equation, limiting somewhat the relevance to real fluid flows. A natural next step would be to seek analogous non-singular solutions, i.e. configurations with finite-area vortices. The stability properties of the point vortex configurations could also be investigated. Actually, since even the Kármán configuration is unstable, these new configurations are doubtless unstable as well; however the dominant unstable modes would be of interest. A generalization of the Kármán wake drag formula to these more general translating configurations might be useful in specialized settings, such as the study of fish propulsion. Indeed it is tempting to think that these configurations have an analogue in axisymmetric periodic vorticity distributions, such as might be found in the wake of a swimming jellyfish. Finally, the connection if any between these new solution polynomials and solutions to integrable dynamical systems, in analogy with that found between multisoliton solutions to the KdV hierarchy and the solution polynomials for the non-periodic case (Adler & Moser 1978), remains to be investigated.

REFERENCES

Adler, M. & Moser, J. 1978 On a class of polynomials connected with the Korteweg-deVries equation. *Commun. Math. Phys.* **61**, 1–30.

Aref, H., Newton, P. K., Stremler, M., Tokieda, T. & Vainchtein, D. 2003 Vortex crystals. *Adv. Appl. Mech.* **39**, 1–79.

Aref, H., Stremler, M. A. & Ponta, F. L. 2006 Exotic vortex wakes – point vortex solutions. *J. Fluids Struct.* **22**, 929–940.

Aref, H. & Van Buren, M. 2005 Vortex triple rings. *Phys. Fluids* **17**, 057104.

Bartman, A. B. 1983 A new interpretation of the Adler-Moser KdV polynomials: Interaction of vortices. In *Nonlinear and Turbulent Processes in Physics* (ed. R. Z. Sagdeev), vol. 3, pp. 1175–1181. Harwood Academic Publishers.

Burchnall, J. L. & Chaundy, T. W. 1930 A set of differential equations which can be solved by polynomials. *Proc. Lond. Math. Soc.* 2, **30**, 401–414.

Kadtke, J. & Campbell, L. 1987 Method for finding stationary states of point vortices. *Phys. Rev.* A **36**, 4360–4370.

Loutsenko, I. 2003 Integrable dynamics of charges related to bilinear hypergeometric equation. *Commun. Math. Phys.* **242**, 251–275.

Loutsenko, I. 2004 Equilibrium of charges and differential equations solved by polynomials. *J. Phys.* A*: Math. Gen.* **37**, 1309–1322.

Montaldi, J., Soulière, A. & Tokieda, T. 2003 Vortex dynamics on a cylinder. *SIAM J. Appl. Dyn. Syst.* **2**, 417–430.

O'Neil, K. 1987 Symmetric configurations of vortices. *Phys. Lett.* A **124** 503–507.

O'Neil, K. 2006 Minimal polynomial systems for point vortex equilibria. *Physica* D **219**, 69–79.

Stremler, M. 2003 Relative equilibria of singly periodic point vortex arrays. *Phys. Fluids* **15**, 3767–3775.

Tkachenko, V. K. 1964 Thesis, Institute of Physical Problems, Moscow.

Wang, Z. J. 2005 Dissecting insect flight. *Annu. Rev. Fluid Mech.* **37**, 183–210.